Midterm paper 2015

1. (a) If $\{f_n\}$ is a sequence of entire functions on $\mathbb{C}$, and f is another function on $\mathbb{C}$ such that f_n converges uniformly to f on every compact subset of $\mathbb{C}$, show that f is entire. (10 points) (You may Morera's theorem without proof, but then you should give a precise statement of that theorem.)

   (b) Let

$$g(z) = \sum_{n=-\infty}^{\infty} e^{-2\pi n^2} e^{2\pi i n z}.$$

   Show that g defines an entire function on $\mathbb{C}$, and that the order of growth of g is less than or equal to 2. (12 points) (Hint: Show first that $-2n^2 + 2|n||z| \leq -n^2 + |z|^2$.)

2. Suppose f is a holomorphic function on the strip S_2, and that there exists a constant $A \geq 0$ such that

$$|f(z)| \leq \frac{A}{1+|z|^2} \quad \text{for all } z \in S_2.$$

   (a) Let L_1 and L_2 be the contours given by the horizontal lines

$$L_1 = \{\operatorname{Im} z = -1\} \quad \text{and} \quad L_2 = \{\operatorname{Im} z = 1\},$$

   both oriented such that the real part of z increases along the contour. Show that

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi i z} - 1} dz - \int_{L_2} \frac{f(z)}{e^{2\pi i z} - 1} dz.$$

   (10 points)

   (b) From part (a), sketch a proof that

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n),$$

   where $\widehat{f}(n) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx$. (14 points) (Hint: Treat the integrals along L_1 and L_2 separately. You should explain why the treatment for the two contour integrals is different.)

3. For each of the following statements, determine whether it is true or false. Justify your answer. (10 points each)

   (a) If f is a holomorphic function on S_2, and there exists a constant $A \geq 0$ such that

$$|f(z)| \leq \frac{A}{1+|z|^{2015}} \quad \text{for all } z \in S_2,$$

   then for any $n \in \mathbb{N}$, there exists a constant B, depending only on A and n, such that

$$|f^{(n)}(z)| \leq \frac{B}{1+|z|^{2015}} \quad \text{for all } z \in S_1.$$

(b) If f is a holomorphic function on the strip S_1, and $f(x)$ is real whenever $x \in [0, 1]$, then $f(x)$ is real for all $x \in \mathbb{R}$.

4. In this question log refers to the principal branch of the logarithm. In particular, it is just the natural logarithm when applied to a positive number.

(a) Is there an entire function on $\mathbb{C}$ whose zero set is precisely $\{\log n \colon n \in \mathbb{N}\}$? If yes, give a construction; if not, explain why not. (10 points)

(b) Repeat part (a) if we replace "an entire function" by "an entire function of finite order". (12 points)

(c) Characterize all entire functions f that are of finite order, and satisfy

$$f(\log n) = n \quad \text{for all } n \in \mathbb{N}.$$

(12 points) (Hint: Use your solution to part (b).)